Partial C^0-estimate for Kähler-Einstein metrics

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1 Introduction

In this paper, we give a proof of my conjecture on the partial C^0-estimate for Kähler-Einstein metrics with positive scalar curvature. As a corollary, as I already pointed out in [Ti09], the Gromov-Hausdorff limits of Kähler-Einstein metrics are projective varieties.

Let M be a compact manifold which admits Kähler metrics with its first Chern class $c_1(M)$ positive. We denote by $\mathcal{K}(M)$ the set of all Kähler metrics ω on M with Kähler class $[\omega] = c_1(M)$. Consider its set

$$\mathcal{K}(M, t_0) \,=\, \{\,\omega \in \mathcal{K}(M)\,|\,\mathrm{Ric}(\omega)\,\geq\, t_0\omega\,\}.$$

Clearly, $\mathcal{K}(M, t_0)$ is empty unless $t_0 \leq 1$ and $\mathcal{K}(M, 1)$ is the set of Kähler-Einstein metrics on M with Kähler class $c_1(M)$.

By the Kodaira embedding theorem, for ℓ sufficiently large, any basis of $H^0(M, K_M^{-\ell})$ embeds M into a projective space $\mathbb{C}P^N$. For any $\omega \in \mathcal{K}(M, t_0)$, choose a Hermitian metric h with ω as its curvature form and any orthonormal basis $\{S_i\}_{0\leq i\leq N}$ of each $H^0(M, K_M^{-\ell})$ with respect to the induced inner product by h and ω. Put

$$\rho_{\omega,\ell}(x) \,=\, \sum_{i=0}^{N} ||S_i||_h^2(x). \tag{1.1}$$

This is independent of the choice of h and the orthonormal basis $\{S_i\}$.

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Conjecture 1.1. *[Ti90] There are uniform constants $c_k = c(k,n) > 0$ for $k \geq 1$ and $\ell_i \to \infty$ such that for any $\omega \in \mathcal{K}(M,t_0)$ and $\ell = \ell_i$ for each i,*

$$\rho_{\omega,\ell} \geq c_\ell > 0. \tag{1.2}$$

Remark 1.2. *In fact, I expect the stronger version of Conjecture 1.1: There are uniform constants $c_k = c(k,n) > 0$ for $k \geq 1$ and $\ell_0 = \ell_0(n)$ such that for any $\omega \in \mathcal{K}(M,t_0)$, and $\ell \geq \ell_0$, $\rho_{\omega,\ell} \geq c_\ell$.*

The resolution of this conjecture will lead to a proof of the Yau-Tian-Donaldson conjecture: If M is K-stable for all sufficiently large ℓ, then M admits a Kähler-Einstein metrics.

If ω_i is a sequence of Kähler metrics on M with $[\omega_i] = c_1(M)$ and their Ricci curvature greater than or equal to $t_0 > 0$, then by taking a subsequence if necessary, we may assume that (M, ω_i) converge to a length space (M_∞, d_∞). On the other hand, for ℓ sufficiently large, we have embeddings $\sigma_i : M \hookrightarrow \mathbb{C}P^N$ by an orthonormal basis of $H^0(M, K_M^{-\ell})$ with respect to ω_i. By taking a subsequence if necessary, we may assume that $\sigma_i(M) \subset \mathbb{C}P^N$ converge to a holomorphic cycle $\bar{M}_\infty \subset \mathbb{C}P^N$. It was known (see [Ti09]) that the irreducibility of $\bar{M}_\infty$ implies Conjecture 1.1.

I have expected since early 90's:

Conjecture 1.3. *The Gromov-Hausdorff limit M_∞ coincides with the complex limit $\bar{M}_\infty$. In particular, $\bar{M}_\infty$ is irreducible*

Our main theorem of this paper is to confirm Conjecture 1.1 for $\mathcal{K}(M,1)$, precisely,

Theorem 1.4. *There are a positive constant $\epsilon = \epsilon(n) > 0$ and sufficiently large $\ell = \ell(n)$ such that $\rho_{\omega,\ell} \geq \epsilon$ for all $\omega \in \mathcal{K}(M,1)$.*

It is known that for each n, there are only finitely many family of compact Kähler manifolds of complex dimension n and with positive first Chern class. Hence, Theorem 1.4 holds for all Kähler-Einstein manifolds of dimension n.

Let (M_i, ω_i) be any sequence of Kähler-Einstein manifolds with $\mathrm{Ric}(\omega_i) = \omega_i$ and which converges to (M_∞, d_∞) in the Gromov-Hausdorff topology. Theorem 1.4 follows from the following:

Theorem 1.5. *There are a positive constant $\epsilon = \epsilon(n) > 0$ and sufficiently large $\ell = \ell(n)$ such that $\rho_{\omega_i,\ell} \geq \epsilon$ for all (M_i, ω_i).*

It follows from [CCT95] that there is a closed subset $S \subset M_\infty$ of Hausdorff codimension at least 4 such that $M_\infty \backslash S$ is a smooth Kähler manifold and d_∞ is induced by a Kähler-Einstein metric ω_∞ outside S with $\mathrm{Ric}(\omega_\infty) = \omega_\infty$. Moreover, ω_i converges to ω_∞ in the C^∞-topology outside S.

A consequence of the above theorem implies (as indicated in [Ti09])

Theorem 1.6. *The Gromov-Hausdorff limit M_∞ is a variety embedded in some $\mathbb{C}P^N$ and S is a subvariety.*

This theorem affirms Conjecture 1.3 for Kähler-Einstein metrics. The proof of this theorem is based on the same arguments as those in the proof of Theorem 1.4, that is, constructing holomorphic sections which separate points for the limit of embeddings σ_i. We will omit the details in this note.

We can extend Theorem 1.5 to the case of almost Kähler-Einstein metrics. Let ω_i be a sequence of Kähler metrics on M with $[\omega_i] = c_1(M)$ and $\mathrm{Ric}(\omega_i) \geq (1-\epsilon_i)\omega_i$, where $\epsilon_i \geq 0$ and $\lim \epsilon_i = 0$. Assume that (M, ω_i) converge to a metric space (M_∞, d_∞) in the Gromov-Hausdorff topology. It is proved in [TW11] that M_∞ is smooth outside a closed subset S of Hausdorff codimension at least 4 and the restriction of d_∞ to $M_\infty \backslash S$ is given by a Kähler-Einstein metric ω_∞. We can prove the following

Theorem 1.7. *Let $(M_\infty, \omega_\infty)$ be as above. Then there is a sufficiently large ℓ such that $\rho_{\omega_\infty,\ell} > 0$.*

The proof of this is based on the regularity theory in [TW11] and an extension of the L^2-estimate to singular spaces like (M_∞, ∞).

Theorem 1.4 and 1.6 were announced with an outlined proof in our expository paper for the proceeding of Calabi's 85th birthday edited by Bourguignon, Chen and Donaldson. In the next section of this note, we provide a proof of the first theorem following the arguments in [Ti12]. The second theorem follows easily as we indicated above. During the preparation of this note, we learned that Donaldson and Sun [DS12] also gave an independent and different proof of Theorem 1.4, though the two proofs share some overlapping ideas which appeared in previous works. One can also find a proof of Theorem 1.6 in [DS12]. In Section 3, we give a proof for Theorem 1.7. In Section 4, we discuss the extension of Theorem 1.4 to conic Kähler-Einstein metrics. An outlined proof will be given while details will appear later.

2 Proving Theorem 1.5

The proof of Theorem 1.5 is essentially a localized version of the proof for the following:

Proposition 2.1. *By taking a subsequence if necessary, for each ℓ, we have that $H^0(M_i, K_{M_i}^{-\ell})$ converges to $H^0(M_\infty, K_{M_\infty}^{-\ell})$ as i tends to ∞ in the sense: There are orthonormal bases $\{\sigma_a^i\}_{0\leq a\leq N}$ of $H^0(M_i, K_{M_i}^{-\ell})$ with respect to h_i such that σ_a^i converges to σ_a^∞ ($0 \leq a \leq N$) as i tends to ∞ and $\{\sigma_a^\infty\}$ forms an orthonormal basis of $H^0(M_\infty, K_{M_\infty}^{-\ell})$.*

As in [Ti89], we prove this by using the L^2-estimate for $\bar\partial$-operator and the theory for elliptic equations. Next we recall from [Ti90]:

Lemma 2.2. *For each i and any $\sigma \in H^0(M_i, K_{M_i}^{-\ell})$, we have the following identities:*

$$\Delta_{\omega_i} ||\sigma||^2 = ||\nabla\sigma||^2 - n\ell\, ||\sigma||^2 \tag{2.1}$$

and

$$\Delta_{\omega_i}||\nabla\sigma||^2 \,=\, ||\nabla^2\sigma||^2 \,-\, ((n+2)\,\ell \,-\, 1)\,||\nabla\sigma||^2. \tag{2.2}$$

Corollary 2.3. *There is a uniform constant C_0 such that for any $\sigma \in H^0(M_i, K_{M_i}^{-\ell})$ ($\ell > 0$), we have*

$$\sup_{M_i} ||\sigma|| \quad \leq \quad C_0 \ell^{\frac{n}{2}} \left(\int_{M_i} ||\sigma||^2 \omega_i^n \right)^{\frac{1}{2}} \tag{2.3}$$

$$\sup_{M_i} ||\nabla\sigma|| \quad \leq \quad C_0 \ell^{\frac{n+1}{2}} \left(\int_{M_i} ||\sigma||^2 \omega_i^n \right)^{\frac{1}{2}}. \tag{2.4}$$

It follows from Lemma 2.2 and the standard Moser iteration since the Sobolev constants of (M_i, ω_i) are uniformly bounded due to some results of C. Croke and P. Li (see [Ti87] and its references).

It follows that by taking a subsequence if necessary, we may assume σ_a^i converges to a σ_a^∞ as i tends to ∞. Furthermore, one can show that $\rho_{\omega_i,\ell}$ are uniformly continuous and converge to $\rho_{\omega_\infty,\ell}$ which is also continuous on M_∞.

Thus, in order to prove Theorem 1.5, we only need to show

$$\inf_x \rho_{\omega_\infty,\ell}(x) > 0. \tag{2.5}$$

Since $\rho_{\omega_\infty,\ell}$ is continuous and M_∞ is compact, it suffices to show that for any $x \in M_\infty$, there is $\ell = \ell_x$ such that

$$\rho_{\omega_\infty,\ell}(x) > 0. \tag{2.6}$$

This can be achieved by using the L^2-estimate and the structure results on M_∞ from [CCT95].

According to [CCT95], for any $r_i \mapsto 0$, by taking a subsequence if necessary, we have a tangent cone C_x of $(M_\infty, \omega_\infty)$ at x, where C_x is the limit $\lim_{i\to\infty}(M_\infty, r_i^{-2}\omega_\infty, x)$ in the Gromov-Hausdorff topology, satisfying:
1. C_x is a Kähler cone with vertex o;
2. Each C_x is regular outside a closed subcone S_x of complex codimension at least 2. Such a S_x is the singular set of C_x;
3. There is an natural Kähler Ricci-flat metric g_x on $C_x \backslash S_x$ which is also a cone metric.

Since g_x is a Kähler cone metric, its Kähler form ω_x is equal to $\sqrt{-1}\partial\bar\partial\rho_x^2$ on the regular part of C_x, where ρ_x denotes the distance function from the vertex of C_x, denoted by x for simplicity. In other words, the trivial bundle $L_x = C_x \times \mathbb{C}$ over C_x admits a Hermitian metric $e^{-\rho_x^2}\,|\cdot|^2$ whose curvature is ω_x.

Without loss of generality, we may choose r_i such that $k_i = r_i^{-2}$ are integers.

Now we fix some notations: For any $r > 0$ and $0 < \delta < \epsilon$, we denote by $V(x; \delta, \epsilon, r)$ the set

$$\{sv \in C_x \,|\, s \in (\delta, r), v \in \partial B_1(x, g_x), d(v, S_x \cap \partial B_1(o, g_x)) > \epsilon\},$$

where $B_R(o, g_x)$ denotes the geodesic ball of (C_x, g_x) centered at the vertex and with radius R.

If C_x has isolated singularity, i.e., $\partial B_1(x)$ is smooth, then we can drop ϵ and write

$$V(x;\delta,r) = \{(sv) \in C_x \,|\, s \in (\delta, r), v \in \partial B_1(x)\}.$$

Let k_i be the above sequence such that $(M_\infty, k_i\omega_\infty, x)$ converges to (C_x, g_x, x). By [CCT95], for any given $\delta, \epsilon > 0$, whenever i is sufficiently large, there are diffeomorphism $\phi_i : V(x;\delta,\epsilon,2) \mapsto M_\infty \backslash S$, where S is the singular set of M_∞, satisfying:

(1) $d(x, \phi_i(V(x;\delta,\epsilon,2))) < \delta r_i$ and $\phi_i(V(x;\delta,\epsilon,2)) \subset B_{3r_i}(x)$, where $B_R(x)$ the geodesic ball of $(M_\infty, \omega_\infty)$ with radius R and center at x;

(2) If g_∞ is the Kähler metric with the Kähler form ω_∞ on $M_\infty \backslash S$, then

$$\lim_{i\to\infty} ||r_i^{-2}\phi_i^* g_\infty - g_x||_{C^6(V(x;\delta/2,\epsilon/2,3))} = 0, \tag{2.7}$$

where the norm is defined in terms of the metric g_x.

Lemma 2.4. *There is an integer $a > 0$ such that for i sufficiently large, there are $\delta_i \mapsto 0$ [1] and isomorphisms ψ_i from the trivial bundle $C_x \times \mathbb{C}$ onto $K_{M_\infty}^{-ak_i}$ over $V(x;\delta,\epsilon,2)$ commuting with ϕ_i satisfying:*

$$||\psi_i(1)||^2 = e^{-\rho_x^2} \quad \text{and} \quad ||D\psi||_{C^4} \le \delta_i, \tag{2.8}$$

where $||\cdot||$ denotes the induced norm on $K_{M_\infty}^{-k_i}$ by g_x, D denotes the covariant derivative with respect to the norms $||\cdot||$ and $e^{-\rho_x^2/2}|\cdot|$.

Proof. First we note that by a recent result of Colding-Naber [CN10], the regular part $\mathrm{Reg}(C_x)$ of C_x is geodesically convex, so its universal cover $\widetilde{\mathrm{Reg}}(C_x)$ is a finite cover, say of the order $a \ge 1$. Since g_x is a Kähler Ricci-flat metric on C_x, the induced metric on $\widetilde{\mathrm{Reg}} C_x$ has its holonomy group in $\mathrm{SU}(n)$ and consequently, $K_{C_x}^{-a}$ admits a parallel section. It follows that for i sufficiently large, $K_{M_\infty}^{-a}$ is trivial over $\phi_i(V)$ for any open subset V whose closure is contained in $\mathrm{Reg}(C_x)$. This makes it plausible to construct the isomorphisms ψ_i. For simplicity, we assume $a = 1$ or equivalently, $\mathrm{Reg}(C_x)$ is simply-connected, otherwise, we do the following on its universal cover $\widetilde{\mathrm{Reg}}(C_x)$. We cover $V(x;\delta,\epsilon,2)$ by finitely many geodesic balls $B_{s_\alpha}(y_\alpha)$ $(1 \le \alpha \le N)$ such that the closure of each $B_{2s_\alpha}(y_\alpha)$ is strongly convex and contained in $\mathrm{Reg}(C_x)$. Now we construct ψ_i. For simplicity of notations, we fix a sufficiently large i and write $r = r_i$, $k = k_i$ etc.. First we construct $\tilde\psi_\alpha$ over each $B_{2s_\alpha}(y_\alpha)$. For any $y \in B_{2s_\alpha}(y_\alpha)$, let $\gamma_y \subset B_{2s_\alpha}(y_\alpha)$ be the unique minimizing geodesic from y_α to y. We define $\tilde\psi_\alpha$ as follows: First we define $\tilde\psi_\alpha(1) \in L_i|_{\phi(y_\alpha)}$, where $L_i = K_{M_\infty}^{-k}$, such that $||\psi(1)||^2 = e^{-\rho_x^2(y_\alpha)}$. Next, for any $y \in U_\alpha$, where $U_\alpha = B_{2s_\alpha}(y_\alpha)$, define $\tilde\psi_\alpha : \mathbb{C} \mapsto L_i|_y$ by $\tilde\psi_\alpha(a(y)) = \tau(\phi(y))$, where $a(y)$ is the parallel transport of 1 along γ_y with respect to the standard norm above and $\tau(\phi(y))$ is the parallel transport of $\psi(1)$ along $\phi \cdot \gamma_y$. Clearly, we have the first equation in (2.8). The estimates on derivatives can be

[1] In fact, δ_i can be chosen depending only on $||k_i\phi^* g_\infty - g_x||_{C^6(V(x;\delta/2,\epsilon/2,3))}$.

done as follows: If $a : U_\alpha \mapsto U_\alpha \times \mathbb{C}$ and $\tau : U_\alpha \mapsto \phi^* L|_{U_\alpha}$ are two sections such that $\tilde{\psi}_\alpha(a) = \tau$, then we have the identity: $D\tau = D\tilde{\psi}_\alpha(a) + \tilde{\psi}_\alpha(Da)$, where D denote the covariant derivatives with respect to various norms on line bundles. By the definition, one can easily see that $D\tilde{\psi}_\alpha(y_\alpha) \equiv 0$. To estimate $D\tilde{\psi}_\alpha$ at y, we differentiate along γ_y to get

$$D_T D_X \tau \,=\, D_T(D_X \tilde{\psi}_\alpha(a)) + \tilde{\psi}_\alpha(D_T D_X a),$$

where T is the unit tangent of γ_y and X is a vector field along γ_y with $[T, X] = 0$. Here we have used the fact that $D_T \tilde{\psi}_\alpha = 0$ which follows from the definition. Using the curvature formula, we see that it is the same as

$$k_i \phi^* \omega_\infty(T, X) \tilde{\psi}_\alpha(a) \,=\, D_T(D_X \tilde{\psi}_\alpha(a)) \,+\, \omega_x(T, X) a.$$

Using

$$\lim_{i \to \infty} k_i \phi^* \omega_\infty \,=\, \lambda_i \phi^* \omega_\infty \,=\, \omega_x,$$

we deduce from the above that $D_T(D_X \tilde{\psi}_\alpha(a))$ converges to 0 as i tends to ∞. Since $D_X \tilde{\psi}_\alpha = 0$ at y_α, we see that $||D\tilde{\psi}_\alpha||_{C^0(U)}$ can be made sufficiently small. The higher derivatives can be bounded in a similar way.

Next we want to modify each $\tilde{\psi}_\alpha$ to get the required $\psi = \psi_i$. For any α, β, we set

$$\theta_{\alpha\beta} \,=\, \tilde{\psi}_\alpha^{-1} \circ \tilde{\psi}_\beta : U_\alpha \cap U_\beta \mapsto S^1.$$

Clearly, $\theta_{\alpha\gamma} \,=\, \theta_{\alpha\beta} \cdot \theta_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, so we have a closed cycle $\{\theta_{\alpha\beta}\}$. Using the fact that $K_{M_\infty}^{-k}$ is trivial over $\phi_i(V)$ for any $V \Subset \mathrm{Reg}(C_x)$, one can deduce from the simply-connectedness that this cycle has to be exact, moreover, by replacing each U_α by $B_{s_\alpha}(x_\alpha)$, we can construct $\zeta_\alpha : B_{s_\alpha}(x_\alpha) \mapsto S^1$ satisfying:
1. $\tilde{\psi}_\alpha \cdot \zeta_\alpha = \tilde{\psi}_\beta \cdot \zeta_\beta$ on $B_{s_\alpha}(x_\alpha) \cap B_{s_\beta}(x_\beta)$;
2. $||D\zeta_\alpha||_{C^3(B_{s_\alpha}(x_\alpha))}$ is dominated by $||D\tilde{\psi}_\alpha||_{C^3(U)}$.
Then we can get ψ by setting $\psi = \tilde{\psi}_\alpha \cdot \zeta_\alpha$ on $V(x; \delta, \epsilon, 2) \cap B_{s_\alpha}(x_\alpha)$. It is easy to show that this ψ has all the properties we asked. □

We also have the following extension property:

Lemma 2.5. *Given any $c > 0$, there is a constant C, which may depend on c and the cone C_x* [2]*, such that for any holomorphic function f on $V(x; \delta, \epsilon, 2)$ with $|f|, |df| \le c$ and $f \ge 1$ on $\partial B_1(o, g_x) \cap V(x; \delta, \epsilon, 2)$, we have*

$$f(\sqrt{\delta} v) \,\ge\, 1 \,-\, C\epsilon^2 \log \delta,$$

where $v \in \partial B_1(o, g_x)$ and $d(v, S_x \cap \partial B_1(x, g_x) \ge 8\epsilon^{\frac{1}{n}}$.

[2] It is possible to remove the dependence on C_x' by using known estimates on Green function over manifolds with non=negative Ricci curvature.

Proof. This is proved by using the Green function on $B_1(x, g_x)$ with boundary value 0. Let η be a cut-off function satisfying: $\eta(t) = 0$ for $t \le 1$, $\eta(t) = 1$ for $t \ge 2$ and $|\eta'(t)| \le 1$. Define

$$F(sv) \;=\; \eta(\delta^{-1}s)\,\eta(\epsilon^{-1}d(v, S_x \cap \partial B_1(x, g_x)))\,f(sv),$$

where $s \in (0,1)$ and $v \in \partial B_1(x, g_x)$. Then F vanishes near the singularity of C_x and smooth on $\mathrm{Reg}(C_x)$. Clearly, ΔF supports in $V(x;\delta,\epsilon,2)\backslash V(x;2\delta,2\epsilon,2)$, moreover, for some uniform constant C', we have

$$|\Delta F| \le C' s^{-2}\epsilon^{-2} \text{ on } V(x;\delta,\epsilon,2)\backslash\left(V(x;2\delta,2\epsilon,2)\cup B_{4\delta}(0,g_x)\right)$$

and

$$|\Delta F| \le C'\delta^{-2} \text{ on } B_{4\delta}(o,g_x)\backslash B_\delta(o,g_x).$$

If $G(\cdot,\cdot)$ denotes the Green function on $B_1(o, g_x)$ with boundary value 0, then we have

$$f(y) \;=\; \int_{\partial B_1(o,g_x)} F(z)\frac{\partial G(y,z)}{\partial \nu}dz \;-\; \int_{B_1(o,g_x)} \Delta F(z) G(y,z) dz. \qquad (2.9)$$

Note that for any $y = \sqrt{\delta}v$ as given and $z \in V(x;\delta,\epsilon,2)\backslash V(x;2\delta,2\epsilon,2)$, we have

$$|G(y,z)| \le C''\min\{\delta^{-(n-1)}, s^{-2(n-1)}\}$$

for some constant C''. Since S_x is codimension at least 4 and the dimension of C_x is at least 4, we can deduce from the above estimates

$$\Big|\int_{B_1(o,g_x)} \Delta F(z) G(y,z) dz\Big| \;\le\; C\epsilon^2\log\delta.$$

Then the lemma follows easily from (2.9). □

Now we can apply the L^2-estimate to proving (2.6), and consequently, Theorem 1.5. Fix $0 < \delta < \epsilon$ small and to be determined later. For each i sufficiently large, there is a section $\tau_i = \psi_i(e)$ of $K_{M_\infty}^{-k_i}$ on $\phi_i(V(x;\delta,\epsilon,2))$ such that $||\tau_i||^2 \;=\; e^{1-\rho_x^2}$. Clearly, $||\tau_i||$ is greater than 1 inside $B_1(x)$ and less than 1 outside $B_1(x)$. By Lemma 2.4, $||\bar\partial\tau_i|| \le C\delta_i$ for some uniform constant C. Let η be the cut-off function satisfying: $\eta(t) = 0$ for $t \le 1$, $\eta(t) = 1$ for $t \ge 2$ and $|\eta'(t)| \le 1$. We define for any $y = sv \in V(x;\delta,\epsilon,2)$

$$\tilde\tau_i(\phi_i(y)) \;=\; \eta(7/2 - s)\,\eta(\delta^{-1}s)\,\eta(\epsilon^{-1}d(v, S_x\cap\partial B_1(o,g_x)))\,\tau^l(\phi_i(y)). \qquad (2.10)$$

Here l is a large integer which depends only on ϵ. Then by choosing ϵ sufficiently small, one can easily show $\tilde\tau_i$ extends to a Lipschitz section of $K_{M_\infty}^{-lk_i}$ on M_∞ satisfying:

(i) $\tilde\tau_i$ coincides with τ_i^ℓ on $\phi_i(V(x;2\delta,2\epsilon,3/2))$;

(ii) $\int_{M_\infty} ||\bar\partial\tilde\tau_i||^2\omega_\infty^n \;\le\; Cr_i^{2n-2}$, where C denotes a uniform constant.

Note that C, C' et al always denote uniform constants. Set $\ell = lk_i$. By the L^2-estimate for $\bar\partial$-operator[3], we get a section v_i of $K_{M_\infty}^{-\ell}$ such that $\bar\partial v_i = \bar\partial \tilde\tau_i$ and

$$\int_{M_\infty} ||v_i||^2 \omega_\infty^n \le \frac{1}{\ell}\int_{M_\infty} ||\bar\partial\tilde\tau_i||^2\omega_\infty^n \le C r_i^{2n-2}\ell^{-1}.$$

Then $\sigma_i = \tilde\tau_i - v_i$ is a holomorphic section of $K_{M_\infty}^{-\ell}$. By (i),

$$\bar\partial v_i = 0 \quad \text{on} \quad \phi_i(V(x;2\delta,2\epsilon,3/2)),$$

then by applying the standard elliptic estimates, we can get

$$\sup_{\phi_i(V(x;2\delta,2\epsilon,3/2)\cap\partial B_1(o,g_x))} ||v_i||^2 \le C(\epsilon r_i)^{-2n}\int_{M_\infty}||v_i||^2\omega_\infty^n \le C'\epsilon^{-2n}r_i^{-2}\ell^{-1}.$$

Choosing $l = 4C'\epsilon^{-2n}$, we can show $||\sigma_i|| \ge 1/2$ on $\phi_i(V(x;2\delta,2\epsilon,1)\cap\partial B_1(o,g_x))$. On the other hand, it is clear that $||\sigma_i||$ are uniformly bounded, hence, as i tends to ∞, σ_i restricted to $\phi_i(B_{3/2}(o,g_x))\backslash S$ converges to a holomorphic function f on $B_{3/2}(o,g_x)$ with $f \ge 1/2$ on $\partial B_1(o,g_x)$. Then it follows from Lemma 2.5 that $||\sigma_i||(\phi_i(\sqrt{\delta}v) \ge 1/2 - C\epsilon^2\log\delta$ for some $v \in \partial B_1(o,g_x)$.

By applying the second estimate in Corollary 2.3 to σ_i, we get

$$\sup_{M_\infty}||\nabla\sigma_i|| \le C''\ell^{\frac{n+1}{2}}\int_{M_\infty}||\sigma_i||^2\omega_\infty^n \le \tilde C\epsilon^{-n(n+1)}r_i^{-1}.$$

Since the distance $d(x_i,\phi_i(\sqrt{\delta}v))$ is less than $2\sqrt{\delta}r_i$ for i sufficiently large, we deduce from the above estimates

$$||\sigma_i||(x_i) \ge 1/4 - C\epsilon^2\log\delta - 2\tilde C\epsilon^{-n(n+1)}\sqrt{\delta},$$

hence, if we choose $0, \delta < \epsilon$ satisfying: $32\tilde C\sqrt{\delta} = \epsilon^{n(n+1)}$ and $16C\epsilon^2\log\delta < 1$, then $\rho_{\omega_\infty,\ell}(x) > 1/8$. The theorem is proved.

3 Proving Theorem 1.7

Clearly, Theorem 1.7 can be proved by the arguments in proving Theorem 1.5 in last section once we prove the following lemma.

Lemma 3.1. *Let (M_∞,ω_∞) be a compact metric space such that ω_∞ is a smooth Kähler-Einstein metric on $M_\infty\backslash S$ for some closed subset S of Hausdorff codimension at least 4 and is the curvature of a Hermitian metric on $K_{M_\infty}^{-1}$. Then for any smooth section v of $K_{M_\infty}^{-\ell}$ with support outside S, there is a unique section w such that $\bar\partial w = \bar\partial v$ and*

$$\int_{M_\infty}||w||^2\omega_\infty^n \le \frac{1}{\ell}\int_{M_\infty}||\bar\partial v||^2\omega_\infty^n.$$

[3] We can do it directly on M_∞ by using the L^2-estimate on M_i and Proposition 2.1.

Remark 3.2. *For $(M_\infty, \omega_\infty)$ from last section, this lemma is trivially true. This is because (M, ω_i) converge to $(M_\infty \backslash S, \omega_\infty)$ in the smooth topology and we have the L^2-estimate for the $\bar\partial$-operator. However, we do not have such an approximation by smooth manifolds with Ricci curvature bounded from below.*

Proof. We outline a proof here. First we observe the Bochner identity on $\Omega^{0,1}(K_{M_\infty}^{-\ell}$:

$$2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = \nabla\bar\nabla + \bar\nabla\nabla + (2\ell + 3).$$

Thus we can minimize the functional

$$E(u) = \int_{M_\infty} \left(||\nabla u||^2 + ||\bar\nabla u||^2 + (2\ell+3)||u||^2 + 4(\bar\partial v, u)\right) \omega_\infty^n$$

among all the sections with finite $H^{1,2}$-norm. It is easy to show that the minimizer, say u, exists and satisfies the equation

$$\bar\partial\bar\partial^* u + \bar\partial^*\bar\partial u = \bar\partial v \quad \text{on} \quad M_\infty \backslash S,$$

moreover, such a minimizer u is smooth outside S and

$$\int_{M_\infty} \left(||\bar\partial u||^2 + ||\bar\partial^* u||^2 + ||u||^2\right) \omega_\infty^n \le \int_{M_\infty} ||\bar\partial v||^2 \omega_\infty^n.$$

Put $w = \bar\partial^* u$, then we have

$$(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) w = \bar\partial^*\bar\partial v.$$

It follows

$$-\Delta ||w|| \le C||w|| + ||\bar\partial^*\bar\partial v||.$$

On the other hand, we have

$$\int_{M_\infty} ||w||^2 \omega_\infty^n < \infty.$$

Hence, by the standard Moser iteration, we can show that $||w||$ is bounded on M_∞. Then we can do integration by parts and get from the equation on w

$$\int_{M_\infty} ||w||^2 \omega_\infty^n \le \frac{1}{\ell} \int_{M_\infty} ||\bar\partial w||^2 \omega_\infty^n \le \frac{1}{\ell} \int_{M_\infty} ||\bar\partial v||^2 \omega_\infty^n.$$

The lemma is proved. □

If $(M_\infty, \omega_\infty)$ is the Gromov-Hausdorff limit of (M, ω_i) with $[\omega_i] = c_1(M)$ and $\mathrm{Ric}(\omega_i) \ge t_i \omega_i$ for some $t_i \to 1$, then we hope to deduce from Theorem 1.7 that $\rho_{\omega_i,\ell} \ge \epsilon$ for a sufficiently large ℓ and a sufficiently small $\epsilon > 0$. We still have the analogous version of Lemma 2.2 and consequently, the estimates in Corollary 2.3. It follows that $\rho_{\omega_i,\ell}$ are uniformly continuous and converge to a continuous limit function on M_∞, however, this limit may not coincide with $\rho_{\omega_\infty,\ell}$ since we do not know an analogue of Proposition 2.1 for almost Kähler-Einstein metrics, or more generally, for Kähler metrics with Ricci curvature bounded from below.

Conjecture 3.3. *For each $t_0 > 0$ and sufficiently large ℓ, $\rho_{\omega,\ell}(x)$ is a uniform continuous function on $(\omega, x) \in \mathcal{K}(M, t_0) \times M$.*

This is a metric version of the flatness for varieties in algebraic geometry. The above conjecture has an important application to establishing the existence of Kähler-Einstein metrics. If Conjecture 3.3 is affirmed, we can deduce from Theorem 1.7

Theorem 3.4. *Let M be a compact Kähler manifold with $c_1(M) > 0$. Assume that the K-energy associated to the Kähler class $c_1(M)$ is bounded from below and (M, K_M^{-1}) is K-stable. Then M admits a Kähler-Einstein metric.*

In fact, we only need a weak version of Conjecture 3.3: In order to prove Theorem 3.4, we only need to prove $\lim \rho_{\omega_i,\ell} = \rho_{\omega_\infty,\ell}$ for a sequence of almost Kähler-Einstein metrics ω_i satisfying: $\mathrm{Ric}(\omega_i) - \omega_i = \sqrt{-1}\partial\bar{\partial} h_i$ with $\lim ||h_i||_{C^0} = 0$.

4 Extension to conic Kähler-Einstein metrics

The theory of smooth Kähler-Einstein metrics can be generalized to the metrics with conic angle along a divisor. For simplicity, here we consider only the case of smooth divisors[4].

Let M be a compact Kähler manifold and $D \subset M$ be a smooth divisor. A conic Kähler metric on M with angle $2\pi\beta$ $(0 < \beta \leq 1)$ along D is a Kähler metric on $M \backslash D$ that is asymptotically equivalent along D to the model conic metric

$$\omega_{0,\beta} \,=\, \sqrt{-1} \left(\frac{d_1 \wedge d\bar{z}_1}{|z_1|^{2-2\beta}} \,+\, \sum_{j=2}^{n} dz_j \wedge d\bar{z}_j \right),$$

where $z_1, z_2, \cdots, z_n$ are holomorphic coordinates such that $D = \{z_1 = 0\}$ locally. Each conic Kähler metric can be given by its Kähler form ω which represents a cohomology class in $H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$, referred as the Kähler class $[\omega]$. A conic Kähler-Einstein metric is a conic Kähler metric which are also Einstein.

In this section, we discuss the generalizations of Theorem 1.4 and Theorem 1.6 to conic Kähler-Einstein metrics of positive scalar curvature. Let us first describe our main results in this section. Let M be a Fano manifold and D be a smooth divisor which represents the Poincare dual of $\lambda c_1(M)$. We call ω a conic Kähler-Einstein if its Kähler class equals to $2\pi c_1(M)$ and satisfies:

$$\mathrm{Ric}(\omega) \,=\, \mu\omega \,+\, (1-\beta)[D]. \tag{4.1}$$

Here the equation on M is in the sense of currents, while it is classical outside D. We require $\mu > 0$ which is equivalent to $(1-\beta)\lambda < 1$. As in the smooth case, each conic Kähler metric ω with $[\omega] = 2\pi c_1(M)$ is the curvature of a Hermitian metric $|| \cdot ||$ on the anti-canonical bundle K_M^{-1}. The difference is that it is

[4]The results in this section still hold for divisors with normal crossings.

not smooth, but it is Hölder continuous. S. Donaldson suggested a continuity method of constructing a Kähler-Einstein metric on M by using conic Kähler-Einstein metrics. It boils down to solving the following complex Monge-Ampere equations:

$$(\omega_\beta + \sqrt{-1}\partial\bar\partial\varphi)^n = e^{h_\beta - \mu\varphi}\omega_\beta^n, \tag{4.2}$$

where ω_β is a suitable family of conic Kähler metrics with $[\omega_\beta] = 2\pi c_1(M)$ and cone angle $2\pi\beta$ along D and h_β is determined by

$$\mathrm{Ric}(\omega_\beta) = \mu\omega + (1-\beta)[D] + \sqrt{-1}\partial\bar\partial h_\beta \quad \text{and} \quad \int_M (e^{h_\beta} - 1)\omega_\beta^n == 0.$$

As shown in [JMR11], if $\mu > 0$ is sufficiently small, (4.2) is solvable, so there is a conic Kähler-Einstein metric with corresponding cone $2\pi\beta$ along D. Furthermore, as shown in [JMR11], it is crucial in solving (4.2) to establish the a prior C^0-estimate for its solutions. Such a C^0-estimate does not hold in general. Therefore, as shown in my program on the existence of Kähler-Einstein metrics through the standard continuity method [Ti09], we first establish a partial C^0-estimate and then use the K-stability to conclude the C^0-estimate, consequently, the existence of Kähler-Einstein metrics on Fano manifolds which are K-stable.

For any $t_0 > 0$ and $\beta_0 > 0$, let $\mathcal{K}(M, D, t_0, \beta_0)$ be the set of conic Kähler metrics with Ricci curvature bounded from below by $t_0 > 0$ and cone angle $2\pi\beta$ along D for some $1 \geq \beta \geq \beta_0$. For any $\omega \in \mathcal{K}(M, t_0)$, choose a C^1-Hermitian metric h with ω as its curvature form and any orthonormal basis $\{S_i\}_{0\leq i\leq N}$ of each $H^0(M, K_M^{-\ell})$ with respect to the induced inner product by h and ω. As before, we have a well-defined function

$$\rho_{\omega,\ell}(x) = \sum_{i=0}^N ||S_i||_h^2(x). \tag{4.3}$$

Conjecture 4.1. *There are uniform constants $c_k = c(k, n, t_0, \beta_0) > 0$ for $k \geq 1$ and $\ell_i \to \infty$ such that for any $\omega \in \mathcal{K}(M, D, t_0, \beta_0)$ and $\ell = \ell_i$ for each i,*

$$\rho_{\omega,\ell} \geq c_\ell > 0. \tag{4.4}$$

The following confirms Conjecture 4.1 for conic Kähler-Einstein metrics with angle bounded from below.

Theorem 4.2. *There are a positive constant $\epsilon = \epsilon(n, \beta_0) > 0$ and sufficiently large $\ell = \ell(n, \beta_0)$ such that for any conic Kähler-Einstein metric ω with angle $2\pi\beta > 2\pi\beta_0$, we have $\rho_{\omega,\ell} \geq \epsilon$.*

In the following, we show the main steps in proving Theorem 4.2. As before, it suffices to prove the estimate $\rho_{\omega_i,\ell} \geq \epsilon$ uniformly for any sequence of conic Kähler-Einstein metrics $\omega_i \in \mathcal{K}(M, D, t_0, \beta_0)$ converging to a metric space (M_∞, d_∞) in the Gromov-Hausdorff topology, furthermore, we have

$$\mathrm{Ric}(\omega_i) = \mu_i\omega_i + (1-\beta_i)[D]$$

with $\lim \mu_i = \mu_\infty > 0$ and $\lim \beta_i = \beta_\infty \geq \beta_0$.

First we establish the followings:

(F1) There is a uniform constant $C = C(M, D, t_0, \beta_0)$ such that for any $\omega_i \in \mathcal{K}(M, D, t_0, \beta_0)$ and function f on M, we have the Sobolev inequality:

$$\left(\int_M |f|^{\frac{2n}{n-1}} \omega^n\right)^{\frac{n-1}{n}} \leq C \left(\int_M (|\nabla f|^2 + |f|^2)\omega^n\right).$$

Applying this to (2.1) and (2.2) and the standard Moser iteration, we still have the following estimates for $\omega_i \in \mathcal{K}(M, D, t_0, \beta_0)$: There is a uniform constant C_0 such that for any $\sigma \in H^0(M_i, K_{M_i}^{-\ell})$ $(\ell > 0)$, we have

$$\sup_M \left(||\sigma|| + \ell^{-\frac{1}{2}} ||\nabla \sigma||\right) \leq C_0 \ell^{\frac{n}{2}} \left(\int_M ||\sigma||^2 \omega^n\right)^{\frac{1}{2}}. \tag{4.5}$$

This can be proved by the same arguments in proving corresponding ones for smooth Kähler-Einstein metrics.

(F2) There is a closed subset $S \subset M_\infty$ of Hausdorff codimension at least 2 such that $M_\infty \backslash S$ is a smooth Kähler manifold and d_∞ is induced by a Kähler-Einstein metric ω_∞ outside S which satisfies

$$\mathrm{Ric}(\omega_\infty) = \mu_\infty \omega_\infty + (1 - \beta_\infty)[D].$$

Moreover, ω_i converges to ω_∞ in the C^∞-topology outside S.

(F3) For any $r_i \mapsto 0$, by taking a subsequence if necessary, $(M_\infty, r_i^{-2}\omega_\infty, x)$ converges to a tangent cone C_x at x satisfying:
1. C_x is a Kähler cone with vertex o;
2. Each C_x is regular outside a closed subcone S_x of complex codimension at least 1. Such a S_x is the singular set of C_x;
3. There is an natural Kähler Ricci-flat metric g_x whose Kähler form ω_∞ is $\sqrt{-1}\partial\bar{\partial}\rho_x^2$ on $C_x \backslash S_x$ which is also a cone metric, where ρ_x denotes the distance function from the vertex of C_x;
4. For any $1 \leq m \leq n$, define S_m to be the set of all points in S which have a tangent cone of the form $\mathbb{C}^{n-m} \times C'_x$, where C'_x admits no line. Then S_m has complex codimension at least m and S is the union of all S_m. Set $S' = \bigcup_{m \geq 2} S_m$, then for any $x \in S \backslash S'$, every tangent cone is of the form $\mathbb{C}^{n-1} \times C_\beta$, where C_β denotes the 2-dimensional flat cone of the angle $2\pi\beta_\infty$;

Both **(F2)** and **(F3)** can be proved by using the techniques developed in [CCT95], but one needs new inputs in the proof which will appear in my joint paper with Z.L. Zhang [TZ12].

(F4) The L^2-estimate holds on M_∞: For any $\tau \in \Lambda^{0,1}(K_{M_\infty}^{-\ell})$[5] with $\bar{\partial}\tau = 0$ and $\int_{M_\infty} ||\tau||^2 \omega_\infty^n < \infty$, where $||\cdot||$ denotes a norm on $K_{M_\infty}^{-1}$ with curvature ω_∞,

[5] This is understood as a section on the regular part of M_∞.

there is a $\sigma \in K_{M_\infty}^{-\ell}$ satisfying:

$$\partial\sigma = \tau \quad \text{and} \quad \int_{M_\infty} ||\sigma||^2 \omega_\infty^n \leq \frac{1}{\ell+\mu} \int_{M_\infty} ||\tau||^2 \omega_\infty^n.$$

This can be proved by applying the L^2-estimate for $\bar{\partial}$-operator to conic manifolds (M, D, ω_i) and taking the limit. This last step is similar to that in the proof of Proposition 2.1.

Once we establish these, the remaining crucial ingredient is to have a version of Lemma 2.4. Then we can proceed as in the proof of Theorem 1.5 except that we need to choose different cut-off functions which correspond to the potential of the Poincare metric on a punctured disc. The details will be presented later.

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